

# Stability of discrete dark solitons in nonlinear Schrödinger lattices

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## Abstract

We obtain new results on the stability of discrete dark solitons bifurcating from the anti-continuum limit of the discrete nonlinear Schrödinger equation, following the analysis of our previous paper [Physica D **212**, 1-19 (2005)]. We derive a criterion for stability or instability of dark solitons from the limiting configuration of the discrete dark soliton and confirm this criterion numerically. We also develop detailed calculations of the relevant eigenvalues for a number of prototypical configurations and obtain very good agreement of asymptotic predictions with the numerical data.

In this paper, we address the dynamical lattice model governed by the discrete nonlinear Schrödinger (DNLS) equation [5]. We consider the defocusing version of this equation in the form

$$i\dot{u}_n + \epsilon(u_{n+1} - 2u_n + u_{n-1}) - |u_n|^2 u_n = 0, \quad (1)$$

where  $n \in \mathbb{Z}$ ,  $u_n(t) : \mathbb{R} \rightarrow \mathbb{C}$ , and  $\epsilon > 0$ . The stationary solutions  $u_n(t) = \phi_n e^{-it}$  are found from second-order difference equation

$$(\phi_n^2 - 1)\phi_n = \epsilon(\phi_{n+1} - 2\phi_n + \phi_{n-1}) \quad (2)$$

for a real-valued sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$ , denoted in vector notations by  $\phi$ . Our aim here is to study discrete *dark* solitons which are defined by the non-vanishing boundary conditions at infinity, e.g.  $\lim_{n \rightarrow \pm\infty} \phi_n = \pm 1$ . The limiting configuration of dark solitons at  $\epsilon = 0$  is defined by the decomposition  $\mathbb{Z} = U_+ \cup U_- \cup U_0$  such that  $\phi_n = \pm 1$  for  $n \in U_{\pm}$  and  $\phi_n = 0$  for  $n \in U_0$ . Our previous work [10] addressed stability of discrete *bright* solitons when  $\dim(U_+ \cup U_-) < \infty$ . In this paper, we shall study stability of discrete dark solitons, when  $\dim(U_0) < \infty$  and there exists  $N \geq 1$  such that  $n \in U_{\pm}$  for all  $\pm n \geq N$ . These solutions were considered recently in [3, 12], as well as earlier in [4, 6] using predominantly numerical computations.

The topic of dark solitons and their stability is not only of theoretical and mathematical interest, but is also a subject of relevance to presently available experimental settings. In particular, discrete dark solitons have been observed in the context of AlGaAs waveguide arrays in the anomalous diffraction regime [8]. Furthermore, as was illustrated in [3], similar phenomenology can be observed

in the discrete dark solitons that arise in defocusing lithium niobate waveguide arrays which exhibit a saturable nonlinearity due to the photovoltaic effect; in the latter case, experimental results were presented in the work of [11]. Although these nonlinear optics experiments are the most promising realizations of discrete dark solitons, such waveforms may also be relevant to the atomic physics. In particular, dark solitons were considered for Bose-Einstein condensates in the presence of a periodic, so-called optical lattice, potential [1, 9] (although in the latter setting, discrete dark solitons have not yet been experimentally realized).

To determine the persistence and stability of discrete dark solitons, we study spectra of the linear operators  $L_+$  and  $L_-$  defined by

$$\begin{aligned}(L_+\psi)_n &= (3\phi_n^2 - 1)\psi_n - \epsilon(\psi_{n+1} - 2\psi_n + \psi_{n-1}), \\ (L_-\psi)_n &= (\phi_n^2 - 1)\psi_n - \epsilon(\psi_{n+1} - 2\psi_n + \psi_{n-1}).\end{aligned}$$

If  $\phi \in l^\infty(\mathbb{Z})$  for any  $\epsilon \geq 0$ , then the operators  $L_\pm$  map  $l^2(\mathbb{Z})$  to itself. Their spectrum at  $\epsilon = 0$  is computed explicitly. The operator  $L_+$  has an eigenvalue 2 of multiplicity  $\dim(U_+) + \dim(U_-) = \infty$  and an eigenvalue  $-1$  of multiplicity  $\dim(U_0) < \infty$ . The operator  $L_-$  has an eigenvalue 0 of multiplicity  $\dim(U_+) + \dim(U_-) = \infty$  and an eigenvalue  $-1$  of multiplicity  $\dim(U_0) < \infty$ .

Since  $l^2(\mathbb{Z})$  is a Banach algebra with respect to the pointwise multiplication and the operator  $L_+$  is continuously invertible in  $l^2(\mathbb{Z})$  for sufficiently small  $\epsilon \geq 0$ , persistence of solutions of the difference equation (2) in  $l^2(\mathbb{Z}) \subset l^\infty(\mathbb{Z})$  with respect to small parameter  $\epsilon$  is proved using the Implicit Function Theorem. Analysis of the stability problem

$$(L_+\mathbf{u})_n = -\lambda w_n, \quad (L_-\mathbf{w})_n = \lambda u_n \quad (3)$$

for small  $\epsilon \geq 0$  is, however, more complicated because of the splitting of the zero eigenvalue of infinite multiplicity into a spectral band located at

$$\Lambda_s = \left\{ \lambda \in \mathbb{C} : -2\sqrt{2\epsilon(1+2\epsilon)} \leq \text{Im}\lambda \leq 2\sqrt{2\epsilon(1+2\epsilon)} \right\}$$

and a number of isolated eigenvalues around the origin. We shall count these eigenvalues by using the recent results of [2, 10].

Since  $(L_-\phi)_n = 0$  and the non-decaying sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$  does not oscillate as  $n \rightarrow \pm\infty$ , 0 is at the bottom of the continuous spectrum of  $L_-$ , which is located for  $\lambda \in [0, 4\epsilon]$ . By the discrete Sturm theory [7], the number of negative eigenvalues of  $L_-$  equals the number of times the sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$  changes sign on  $\mathbb{Z}$ . To compute this number, we subdivide  $U_0$  into segments  $U_0 = \cup_{j=1}^N [n_j^-, n_j^+]$  for some  $N < \infty$  and denote the number of sign-changes between adjacent nodes in  $U_+ \cup U_-$  by  $\sigma_0$ .

**Lemma 1** *There exists  $\epsilon_0 > 0$  such that, for any  $\epsilon \in (0, \epsilon_0)$ , the number of sign changes of the sequence  $\{\phi_n\}_{n \in \mathbb{Z}}$  equals  $\dim(U_0) + \sigma_0 + \sum_{j=1}^N \sigma_j$ , where  $\sigma_j$  is associated with the segment  $U_j = [n_j^-, n_j^+] \subset U_0$ , such that*

$$\sigma_j = \begin{cases} 1 & \text{if } \dim(U_j) \text{ is odd and } \text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = 1 \\ 0 & \text{if } \dim(U_j) \text{ is odd and } \text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = -1 \\ 1 & \text{if } \dim(U_j) \text{ is even and } \text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = -1 \\ 0 & \text{if } \dim(U_j) \text{ is even and } \text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = 1 \end{cases} \quad (4)$$

**Proof.** By persistence of solutions in  $l^\infty(\mathbb{Z})$ -norm for sufficiently small  $\epsilon$ , any sign change between the adjacent nodes in  $U_+ \cup U_-$  persists in  $\epsilon$ . Therefore, the statement of the lemma is proved if we can prove for a particular segment  $U_j = [n_j^-, n_j^+]$  of length  $n_j = n_j^+ - n_j^- + 1$  that the number of sign changes equals  $n_j + \sigma_j$ , where  $\sigma_j$  is given by (4). To do this with an application of Lemma 2.3 from [10], we use the staggering transformation  $\phi_n = (-1)^n \varphi_n$  and rewrite the different equation (2) in the form

$$(1 - \varphi_n^2) \varphi_n = \epsilon (\varphi_{n+1} + 2\varphi_n + \varphi_{n-1}).$$

By Lemma 2.3 of [10], there is only one sign difference in the sequence  $\{\varphi_n\}_{n_j^- - 1}^{n_j^+ + 1}$  if  $\text{sign}(\varphi_{n_j^- - 1} \varphi_{n_j^+ + 1}) = -1$  and none if  $\text{sign}(\varphi_{n_j^- - 1} \varphi_{n_j^+ + 1}) = 1$ . If  $n_j$  is odd, the staggering transformation gives  $(n_j + 1)$  sign differences in the sequence  $\{\phi_n\}_{n_j^- - 1}^{n_j^+ + 1}$  if  $\text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = 1$  and  $n_j$  sign differences if  $\text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = -1$ . If  $n_j$  is even, however, the sequence  $\{\phi_n\}_{n_j^- - 1}^{n_j^+ + 1}$  has  $n_j$  sign differences if  $\text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = 1$  and  $(n_j + 1)$  sign differences if  $\text{sign}(\phi_{n_j^- - 1} \phi_{n_j^+ + 1}) = -1$ . Thus, formula (4) is proved.  $\blacksquare$

**Corollary 2** *The number  $N_0 = \sigma_0 + \sum_{j=1}^N \sigma_j$  equals the number of small negative eigenvalues of  $L_-$  for  $\epsilon > 0$  bifurcating from the zero eigenvalue of infinite multiplicity for  $\epsilon = 0$ .*

**Proof.** This follows from the fact that  $L_-$  has  $\dim(U_0)$  negative eigenvalues at  $\epsilon = 0$ .  $\blacksquare$

**Definition 3** *The stability problem (3) is said to have a purely imaginary eigenvalue of negative Krein signature if  $(L_- \mathbf{u}, \mathbf{u}) = (L_+^{-1} \mathbf{w}, \mathbf{w}) \leq 0$  for the corresponding eigenvector  $(\mathbf{u}, \mathbf{w})$ .*

**Theorem 4** *There exists  $\epsilon_0 > 0$  such that, for any  $\epsilon \in (0, \epsilon_0)$ , the stability problem (3) has exactly  $\dim(U_0)$  pairs of purely imaginary isolated eigenvalues of negative Krein signature bounded away from the continuous spectrum and exactly  $N_0$  pairs of small real eigenvalues.*

**Proof.** Since  $L_+$  is invertible for sufficiently small  $\epsilon$ , we rewrite the stability problem (3) in the form

$$L_- \mathbf{w} = \gamma L_+^{-1} \mathbf{w}, \quad \gamma = -\lambda^2. \quad (5)$$

Since  $L_-$  is not invertible for any  $\epsilon \geq 0$ , we shift the eigenvalue problem to the form

$$(L_- + \delta L_+^{-1}) \mathbf{w} = (\gamma + \delta) L_+^{-1} \mathbf{w},$$

for sufficiently small  $\delta > 0$ . Since  $\text{Null}(L_-)$  lies in the positive subspace of  $L_+^{-1}$  at  $\epsilon = 0$  and the number of negative eigenvalues of  $L_-$  is unchanged in  $\epsilon \in (0, \epsilon_0)$ , for a fixed  $\epsilon \in (0, \epsilon_0)$ , there is a small  $\delta = \delta(\epsilon)$ , such that the number of negative eigenvalues of  $L_- + \delta L_+^{-1}$  is the same as that of  $L_-$ . Conditions of Theorem 1 of [2] are now satisfied and we count the negative eigenvalues of  $L_- + \delta L_+^{-1}$  (same as for  $L_-$ ) and  $L_+^{-1}$  (same as for  $L_+$ ) as follows:

$$n(L_-) = N_p^- + N_n^+ + N_{c^+}, \quad n(L_+) = N_n^- + N_n^+ + N_{c^+},$$

where  $n(L_{\pm})$  denotes the number of negative eigenvalues of  $L_{\pm}$ ,  $N_{c+}$  denotes the number of complex eigenvalues  $\gamma$  in the upper half-plane,  $N_n^{\pm}$  denote the number of positive/negative eigenvalues  $\gamma$  with  $(\mathbf{w}, L_+^{-1}\mathbf{w}) \leq 0$  for corresponding eigenvectors  $\mathbf{w}$ , and  $N_p^-$  denotes the number of negative eigenvalues  $\gamma$  with  $(\mathbf{w}, L_+^{-1}\mathbf{w}) \geq 0$  for corresponding eigenvectors  $\mathbf{w}$ . By Lemma 1 and Corollary 2, we have  $n(L_-) = \dim(U_0) + N_0$  and  $n(L_+) = \dim(U_0)$  for sufficiently small  $\epsilon \in (0, \epsilon_0)$ .

At  $\epsilon = 0$ , there exists  $\dim(U_0)$  eigenvalues  $\gamma = 1$  with  $(\mathbf{w}, L_+^{-1}\mathbf{w}) < 0$ , where the sequence  $\{w_n\}_{n \in \mathbb{Z}}$  for the eigenvector  $\mathbf{w}$  is compactly supported in  $U_0$ . By Proposition 5.1 in [2], the eigenvalue  $\gamma = 1$  is hence semi-simple (that is algebraic and geometric multiplicities coincide). Therefore, all  $\dim(U_0)$  eigenvalues persist for positive values of  $\gamma$  for sufficiently small  $\epsilon$ . By continuity of the eigenvectors  $\mathbf{w}$  in  $\epsilon$ , the inequality  $(\mathbf{w}, L_+^{-1}\mathbf{w}) < 0$  holds for sufficiently small  $\epsilon > 0$ , and therefore,  $n(L_+) = \dim(U_0) = N_n^+$ , such that  $N_{c+} = N_n^- = 0$  and  $N_p^- = N_0$ . Therefore, all  $N_0$  bifurcations of small negative eigenvalues of  $L_-$  for  $\epsilon \in (0, \epsilon_0)$  from the zero eigenvalue of  $L_-$  for  $\epsilon = 0$  result in pairs of small real eigenvalues  $\lambda = \pm\sqrt{-\gamma}$  of the stability problem (3). ■

**Remark 5** *Small eigenvalues of the operator  $L_-$  can be found from the difference eigenvalue problem*

$$V_n \psi_n - \epsilon(\psi_{n+1} + \psi_{n-1} - 2\psi_n) = \mu \psi_n, \quad V_n = V_n^{(0)} + \sum_{k=1}^{\infty} \epsilon^k V_n^{(k)}, \quad (6)$$

where  $V_n^{(0)} = (\phi_n^{(0)})^2 - 1$ ,  $V_n^{(1)} = 2\phi_n^{(0)}\phi_n^{(1)}$ ,  $V_n^{(2)} = 2\phi_n^{(0)}\phi_n^{(2)} + (\phi_n^{(1)})^2$  and so on, due to analytic dependence of the solution  $\phi$  of the difference equation (2) on  $\epsilon$ . If  $\mathbf{w}$  is supported in  $U_+ \cup U_-$  and  $\epsilon = 0$ , then  $L_+\mathbf{w} = 2\mathbf{w}$ . Since  $l^2$ -eigenvectors of the difference equation (6) for small negative eigenvalues  $\mu$  are supported in  $U_+ \cup U_-$  as  $\epsilon \rightarrow 0$ , a small negative eigenvalue  $\mu$  for  $L_-$  is related to a small negative eigenvalue  $\gamma$  for  $L_+L_-$  (that is the eigenvalue of the stability problem (5) with the  $l^2$ -eigenvector) by the asymptotic approximation  $\lim_{\epsilon \rightarrow 0} \gamma/\mu = 2$ .

As the simplest application of our results, we consider two basic configurations of discrete dark solitons from [3, 4, 6].

- If  $U_{\pm} = \mathbb{Z}_{\pm}$  and  $U_0 = \{0\}$  (a so-called *on-site* dark soliton), then  $N_0 = 0$  and, according to Theorem 4, the dark soliton is stable with a single pair of purely imaginary eigenvalues of negative Krein signature near  $\lambda = \pm i$ .
- If  $U_+ = \mathbb{Z}_+$ ,  $U_- = \mathbb{Z}_- \cup \{0\}$ , and  $U_0 = \emptyset$  (a so-called *inter-site* dark soliton), then  $N_0 = \sigma_0 = 1$  and the dark soliton is unstable with a single pair of real eigenvalues. The asymptotic approximation of the unstable eigenvalue can be obtained with the following argument. The solution of the difference equation (2) is expanded in the power series  $\phi = \phi^{(0)} + \epsilon\phi^{(1)} + O(\epsilon^2)$ , where  $\phi^{(1)}$  is compactly supported with  $\phi_0^{(1)} = 1$ ,  $\phi_1^{(1)} = -1$  and  $\phi_n^{(1)} = 0$  for all  $n \in \mathbb{Z} \setminus \{0, 1\}$ . Since  $V_n^{(0)} = 0$  for all  $n \in \mathbb{Z}$  and  $V_n^{(1)} = -2$  for  $n = \{0, 1\}$  and  $V_n^{(1)} = 0$  otherwise, the potential  $V$  of the discrete Schrödinger equation (6) is negative at the leading order. By the discrete Sturm theory [7], it traps a unique negative eigenvalue with the symmetric eigenfunction  $\psi_n = \psi_{-n+1}$ ,  $n \in \mathbb{N}$ . Using the parametrization

$$\mu = \epsilon(2 - e^{\kappa} - e^{-\kappa}) \quad (7)$$

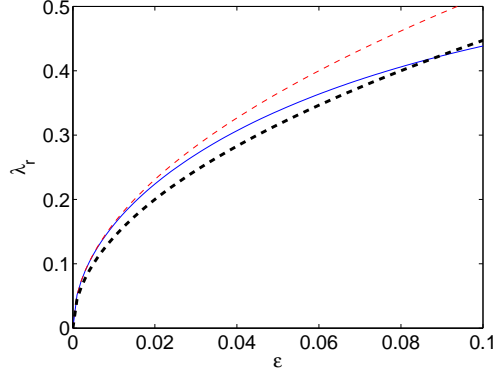


Figure 1: Numerical approximations of the unstable eigenvalue for the inter-site dark soliton (solid line) together with the asymptotic prediction  $\lambda = \sqrt{\frac{8\epsilon}{3}}$  (dashed curve) and the approximation  $\lambda = \sqrt{2\epsilon}$  of [3] (thick dashed curve).

and solving the eigenvalue problem for the eigenvector  $\psi_1 = 1$ ,  $\psi_n = Ce^{-\kappa(n-2)}$  for  $n \geq 2$ , we obtain  $C = e^{-\kappa}$  and  $e^\kappa = 3$  at the leading order of  $O(\epsilon)$ , which gives  $\mu = -\frac{4}{3}\epsilon + O(\epsilon^2)$ . Using Remark 5, we conclude that the pair of real eigenvalues of the stability problem (3) is given by  $\lambda = \pm\sqrt{-\gamma} = \pm\sqrt{\frac{8\epsilon}{3}}(1 + O(\epsilon))$ . This approximation is shown on Fig. 1 with thin dashed line, while the solid line shows results of numerical approximations of eigenvalues of the truncated linear stability system (3). It should be noted that the earlier work of [3] approximated the real eigenvalue pair of the inter-site dark soliton as  $\lambda = \pm\sqrt{2\epsilon}$ . As can be readily observed from a solid dashed line in Fig. 1, this asymptotic prediction is incorrect.

To illustrate more complicated applications of Theorem 4, we consider several composite discrete dark solitons, some of which were studied in [12].

- If  $U_+ = \{0\} \cup \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \emptyset$ , and  $U_- = \mathbb{Z}_- \cup \{1\}$ , then  $N_0 = \sigma_0 = 3$ , such that three pairs of real (unstable) eigenvalues occurs in the linearized problem (3). To find asymptotic approximations of these eigenvalues, we again consider eigenvalues of  $L_-$  in the difference equation (6) with the potentials  $V_n^{(0)} = 0$  for all  $n \in \mathbb{Z}$  and

$$\phi_n^{(1)} = \begin{cases} 1, & n = -1 \\ -2, & n = 0 \\ 2, & n = 1 \\ -1, & n = 2 \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(1)} = 2\phi_n^{(0)}\phi_n^{(1)} = \begin{cases} -4, & n = \{0, 1\} \\ -2, & n = \{-1, 2\}, \\ 0, & \text{otherwise} \end{cases}$$

We construct two symmetric eigenvectors and one anti-symmetric eigenvector for three negative eigenvalues  $\mu$ . For symmetric eigenvectors,  $\psi_n = \psi_{-n+1}$ ,  $n \in \mathbb{Z}$  with  $\psi_1 = 1$ ,  $\psi_2 = B$ ,  $\psi_n = Ce^{-\kappa(n-3)}$ ,  $n \geq 3$ , we use the parametrization (7) and obtain at the leading order of  $O(\epsilon)$ :

$$C = Be^{-\kappa}, \quad (e^\kappa - 2)B = 1, \quad B = e^\kappa + e^{-\kappa} - 5.$$

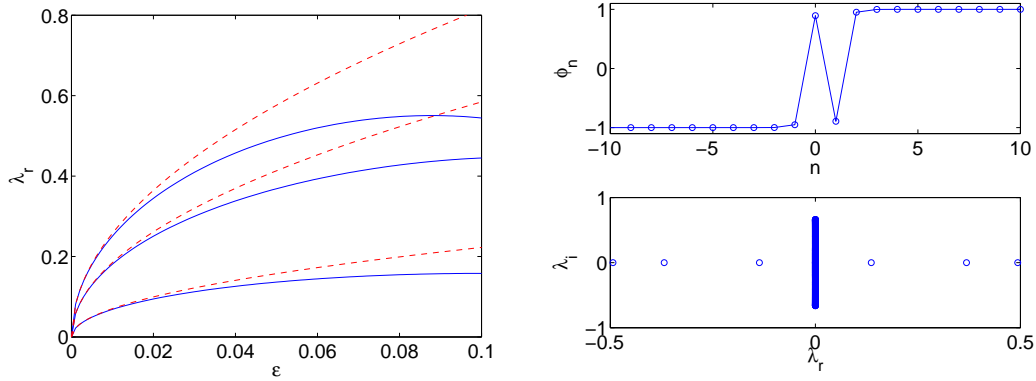


Figure 2: The left panel compares the asymptotic predictions (dashed lines) with the results of numerical linear stability analysis (solid lines) for the three positive real eigenvalues of the discrete dark soliton with  $U_+ = \{0\} \cup \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \emptyset$ , and  $U_- = \mathbb{Z}_- \cup \{1\}$ . The right panel shows a typical example of the solution profile (top) and the corresponding spectral plane  $\lambda = \lambda_r + i\lambda_i$  of its linearization spectrum for  $\epsilon = 0.05$ .

Eliminating  $B$ , we obtain a cubic equation for  $z = e^\kappa$ :

$$z^3 - 7z^2 + 10z - 2 = 0. \quad (8)$$

There exist two solutions of the cubic equation in the interval  $z > 1$ , namely  $z_1 \approx 1.63667$  and  $z_2 \approx 5.12489$ . For the anti-symmetric eigenvector,  $\psi_n = -\psi_{-n+1}$ ,  $n \in \mathbb{Z}$  with  $\psi_1 = 1$ ,  $\psi_2 = B$ ,  $\psi_n = Ce^{-\kappa(n-3)}$ ,  $n \geq 3$ , we obtain at the leading order of  $O(\epsilon)$ :

$$C = Be^{-\kappa}, \quad (e^\kappa - 2)B = 1, \quad B = e^\kappa + e^{-\kappa} - 3.$$

Eliminating  $B$ , we obtain a cubic equation for  $z = e^\kappa$ :

$$z^3 - 5z^2 + 6z - 2 = 0. \quad (9)$$

Since one root is  $z = 1$ , we can find a unique root of the cubic equation in the interval  $z > 1$ , namely  $z_3 = 2 + \sqrt{2}$ . Each of the three roots above generates a negative eigenvalue for  $\mu = \epsilon(2 - z - z^{-1}) + O(\epsilon^2)$ . Each negative eigenvalue  $\mu$  of  $L_-$  generates a negative eigenvalue  $\gamma$  of  $L_+L_-$  with the correspondence  $\gamma = 2\mu + O(\epsilon^2)$ . Summarizing, the three pairs of real eigenvalues are given asymptotically by  $\lambda = \pm 0.70380\sqrt{\epsilon}$ ,  $\lambda = \pm 1.84776\sqrt{\epsilon}$  and  $\lambda = \pm 2.57683\sqrt{\epsilon}$ . These theoretical predictions are compared with the results of full numerical linear stability analysis in the left panel of Fig. 2, yielding a good agreement for small values of  $\epsilon$ . A typical example of the discrete dark soliton and its linearization spectrum for  $\epsilon = 0.05$  is shown in the right panel of Fig. 2.

- If  $U_+ = \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \{0\}$ , and  $U_- = \mathbb{Z}_- \cup \{1\}$ , then  $\sigma_0 = 1$ ,  $\sigma_1 = 1$ , such that  $N_0 = 2$  and two pairs of real (unstable) eigenvalues occur in the linearized problem (3), while one pair of imaginary eigenvalues of negative Krein signature persists on the imaginary axis near  $\lambda = \pm i$ .

To compute the small negative eigenvalues of  $L_-$ , we compute the leading-order potential  $V_n^{(0)} = -1$  for  $n = 0$  and  $V_n^{(0)} = 0$  otherwise and then proceed with the first-order potential:

$$\phi_n^{(1)} = \begin{cases} \frac{1}{2}, & n = -1 \\ 2, & n = 0 \\ \frac{3}{2}, & n = 1 \\ -1, & n = 2 \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(1)} = 2\phi_n^{(0)}\phi_n^{(1)} = \begin{cases} -1, & n = -1 \\ -3, & n = 1 \\ -2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Since the potential has no symmetry, we have to find the eigenvector of the eigenvalue problem (6) in the most general form  $\psi_n = Ae^{\kappa(n+2)}$ ,  $n \leq -2$ ,  $\psi_{-1} = B$ ,  $\psi_0 = C$ ,  $\psi_1 = D$ ,  $\psi_2 = E$ , and  $\psi_3 = Fe^{-\kappa(n-3)}$ ,  $n \geq 3$ . Using the parametrization (7), we obtain at the leading order of  $O(\epsilon)$ :

$$F = Ee^{-\kappa}, \quad (e^\kappa - 2)E = D, \quad C + E = (e^\kappa + e^{-\kappa} - 3)D, \quad A = Be^{-\kappa}, \quad e^\kappa B = B + C.$$

Since the equation at  $n = 0$  implies that  $C = -\epsilon(B + D) + O(\epsilon^2)$ , the chain of equations is uncoupled at the variables  $(D, E, F)$  and  $(A, B)$  at the leading order. Working with the chain for  $(D, E, F)$ , we obtain the same cubic equation (9) with the same root  $e^\kappa = 2 + \sqrt{2}$ , which gives the approximation  $\mu = -\epsilon(1 + \frac{1}{\sqrt{2}}) + O(\epsilon^2)$ . Working with the chain for  $(A, B)$ , we obtain the equation  $e^\kappa = 1$ , which hides a small root  $\kappa = O(\epsilon)$ . To unveil this hidden eigenvalue, we have to extend the potential to the second order by

$$\phi_n^{(2)} = \begin{cases} \frac{1}{4}, & n = \{-2, 2\} \\ \frac{1}{8}, & n = -1 \\ 2, & n = 0 \\ \frac{19}{8}, & n = 1 \\ -\frac{1}{2}, & n = 3 \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(2)} = 2\phi_n^{(0)}\phi_n^{(2)} + (\phi_n^{(1)})^2 = \begin{cases} -\frac{1}{2}, & n = -2 \\ -\frac{3}{2}, & n = -1 \\ 4, & n = 0 \\ -\frac{5}{2}, & n = 1 \\ \frac{3}{2}, & n = 2 \\ -1, & n = 3 \\ 0, & \text{otherwise} \end{cases}$$

Using the parametrization (7) and the representation of the eigenvector in the form  $\psi_n = Ae^{\kappa(n+3)}$ ,  $n \leq -3$ ,  $\psi_{-2} = B$ ,  $\psi_{-1} = C$ ,  $\psi_0 = D$ ,  $\psi_1 = E$ ,  $\psi_2 = F$ ,  $\psi_3 = G$ , and  $\psi_4 = He^{-\kappa(n-4)}$ ,  $n \geq 4$ , we obtain at the leading order of  $O(\epsilon) + O(\epsilon^2)$ :

$$H = Ge^{-\kappa}, \quad (e^\kappa - \epsilon)G = F, \quad E + G = \left(e^\kappa + e^{-\kappa} - 2 + \frac{3\epsilon}{2}\right)F, \quad D + F = \left(e^\kappa + e^{-\kappa} - 3 - \frac{5\epsilon}{2}\right)E,$$

and

$$A = Be^{-\kappa}, \quad (e^\kappa - \frac{\epsilon}{2})B = C, \quad B + D = \left(e^\kappa + e^{-\kappa} - 1 - \frac{3\epsilon}{2}\right)C,$$

where the approximation  $D = -\epsilon(C + E) + O(\epsilon^2)$  is sufficient for the purpose. Eliminating  $B$ ,  $D$ ,  $F$ , and  $G$ , we obtain two equations at the leading order  $O(\epsilon)$ :

$$-\epsilon E = (e^\kappa - 1 - \epsilon)C, \quad -\epsilon C = (2e^\kappa + e^{-\kappa} - 3 - \epsilon)E$$

Using the asymptotic expansion  $\kappa = \epsilon\kappa_1 + O(\epsilon^2)$ , we reduce the problem to a quadratic equation for  $\kappa_1$  with two roots  $\kappa_1 = 2$  and  $\kappa_1 = 0$ . The non-zero root leads to the approximation

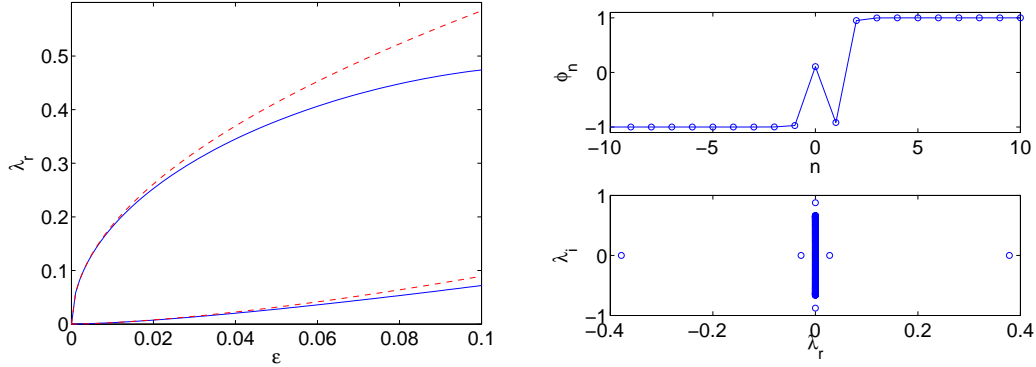


Figure 3: The same as Fig. 2, but for the discrete dark soliton with  $U_+ = \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \{0\}$ , and  $U_- = \mathbb{Z}_- \cup \{1\}$ .

$\mu = -4\epsilon^3 + O(\epsilon^4)$ . Each of the two roots above generates a negative eigenvalue  $\gamma$  of  $L_+L_-$  with the correspondence  $\gamma = 2\mu(1 + O(\epsilon))$ . Summarizing, the two pairs of real eigenvalues are given asymptotically by  $\lambda = \pm 1.84776\sqrt{\epsilon}$  and  $\lambda = \pm\sqrt{8\epsilon^3}$ , which are again found in Fig. 3 to be in very good agreement with the full numerical results.

- If  $U_+ = \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \{0, 1\}$ , and  $U_- = \mathbb{Z}_-$ , then  $N_0 = \sigma_1 = 1$  and one pair of real (unstable) eigenvalues occurs in the linearized problem (3), while two pairs of imaginary eigenvalues of negative Krein signature persist on the imaginary axis near  $\lambda = \pm i$ . To compute the small negative eigenvalue of  $L_-$ , we use Wolfram's MATHEMATICA and compute the potentials of the eigenvalue problem (6) up to the fourth order

$$V_n^{(0)} = \begin{cases} -1, & n = \{0, 1\} \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(1)} = \begin{cases} -1, & n = \{-1, 2\} \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(2)} = \begin{cases} -\frac{1}{2}, & n = \{-2, -1, 2, 3\} \\ 1, & n = \{0, 1\}, \\ 0, & \text{otherwise} \end{cases}$$

and

$$V_n^{(3)} = \begin{cases} -\frac{1}{4}, & n = \{-3, 4\} \\ \frac{1}{8}, & n = \{-2, 3\}, \\ -\frac{21}{8}, & n = \{-1, 2\}, \\ 5, & n = \{0, 1\}, \\ 0, & \text{otherwise} \end{cases}, \quad V_n^{(4)} = \begin{cases} -\frac{1}{8}, & n = \{-4, 5\} \\ \frac{5}{16}, & n = \{-3, 4\}, \\ -\frac{15}{8}, & n = \{-2, 3\}, \\ -\frac{129}{16}, & n = \{-1, 2\}, \\ \frac{45}{2}, & n = \{0, 1\}, \\ 0, & \text{otherwise} \end{cases}$$

Using the parametrization (7) and the symmetry of the eigenvector  $\psi_n = \psi_{-n+1}$ ,  $n \in \mathbb{Z}$  with

$$\psi_1 = \begin{cases} A, & n = 1 \\ B, & n = 2, \\ C, & n = 3, \\ D, & n = 4, \\ E, & n = 5, \\ Fe^{-\kappa(n-6)}, & n \geq 6 \end{cases}$$



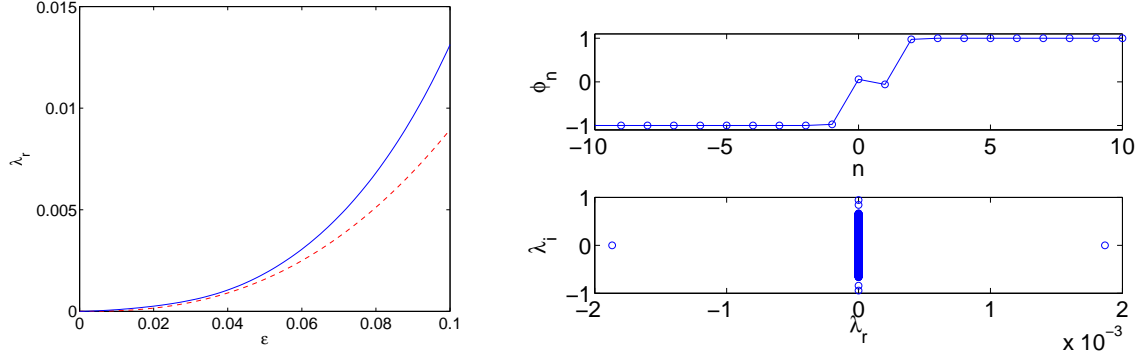


Figure 4: The same as Fig. 2, but for the discrete dark soliton with  $U_+ = \mathbb{Z}_+ \setminus \{1\}$ ,  $U_0 = \{0, 1\}$ , and  $U_- = \mathbb{Z}_-$ .

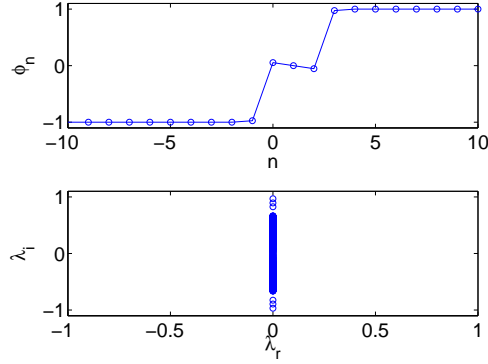


Figure 5: A typical example of the solution profile (top) and the spectral plane of its linearization spectrum (bottom) for the discrete dark soliton with  $U_+ = \mathbb{Z}_+ \setminus \{1, 2\}$ ,  $U_0 = \{0, 1, 2\}$ , and  $U_- = \mathbb{Z}_-$  for  $\epsilon = 0.05$ . As predicted, the configuration is linearly stable for small  $\epsilon$ , bearing three pairs of imaginary eigenvalues (with negative Krein signature), but no real eigenvalue pairs.

we obtain algebraic equations for coefficients  $A \dots F$ , which are solvable up to the fourth order, subject to the characteristic equation  $e^\kappa = 1 + 2\epsilon^2 + O(\epsilon^3)$ . Therefore,  $\kappa = 2\epsilon^2 + O(\epsilon^3)$ , such that  $\mu = -4\epsilon^5 + O(\epsilon^6)$ . The small negative eigenvalue of  $L_+L_-$  is thus approximated by  $\gamma = -8\epsilon^5 + O(\epsilon^6)$ , while the pair of real eigenvalues of the stability problem (3) is given by  $\lambda = \pm\sqrt{8\epsilon^5}(1 + O(\epsilon))$ . The prediction for this small real eigenvalue, leading to a very weak instability in this case, is compared to numerical results in Fig. 4. Once again, we report very good agreement between the two.

- If  $U_+ = \mathbb{Z}_+ \setminus \{1, 2\}$ ,  $U_0 = \{0, 1, 2\}$ , and  $U_- = \mathbb{Z}_-$ , then  $N_0 = 0$  and three pairs of imaginary eigenvalues of negative Krein signature persist on the imaginary axis near  $\lambda = \pm i$ . This is confirmed in Fig. 5, showing a typical example of the discrete dark soliton and its linearization spectrum for  $\epsilon = 0.05$ .

In summary, we have offered a systematic way to assess the stability of discrete dark solitons in the prototypical dynamical lattice model of the DNLS equation. We have illustrated how the number of sign changes in the limiting configuration at the anti-continuum limit can be used to count the number  $N_0$  of small real eigenvalues of its linearization spectrum, when deviating from the anti-continuum limit. We have also associated the number of zeros in the limiting sequence with the number of isolated imaginary eigenvalues of negative Krein signature (which accounts for potential oscillatory instabilities for larger values of the coupling). In addition to this full characterization of the real and imaginary eigenvalues, we have offered a systematic approach towards computing asymptotic approximations of the relevant eigenvalues. In particular, we have connected small eigenvalues of operator  $L_+L_-$  to the small eigenvalues of operator  $L_-$  and have developed perturbation series expansions in terms of the inter-site coupling constant. Within this method, relevant computations result in a finite-dimensional matrix problem. We have demonstrated this approach in a variety of configurations including the on-site and inter-site dark soliton structures of [3, 4, 6], but also in multiple-hole configurations of [12], finding very good agreement between the analytical considerations and the full numerical results. It would be of particular interest to extend relevant computations to higher dimensional settings, examining, for instance, the stability of discrete defocusing vortices in the two- or three-dimensional DNLS equations. Such considerations are deferred to future studies.

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